Advanced Studies in Contemporary Mathematics
34 (2024), No. 1, pp. 19 – 28
www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2024.34.1.19

A MODIFIED CUBIC SPLINE FOR SOLVING INTEGRAL EQUATIONS WITH LOGARITHMIC KERNEL

MOHAMED RAID NADIR AND ADEL JAWAHDOU

ABSTRACT. The main purpose of this work is to present a numerical method for solving a weakly singular integral equations (W.S.I.E) with logarithmic kernel on a piecewise smooth integration path using a modified cubic approximation, this technical is used to convert (W.S.I.E) into a system of algebraic equations and shows the convergence of the approximate solution to the exact one on all points of subduvision of the path. Many examples are given to illustrate the theoretical results and demonstrated the efficiency of this approximation

1. Introduction

Weakly singular integral equations of the second kind appear in many applications transport theory, potential theory and fracture mechanics and elasticity [3]. Noting that, the numerical solution of this kind of equations is used to study the stress field in the vicinity of a crack situated at the interface of two bonded dissimilar half-planes. Many methods are proposed to solve numerically the integral equations of the second kind with logarithmic singular kernel. Cosine and sine Wavelet Method [13], a trigonometric Hermite wavelet approximation is used to solve this integral equations [2], the authors in [9] studied the superconvergence of piecewise polynomial collocations for nonlinear weakly singular integral equations. Prossdorf [10] used a discrete method for the logarithmic kernel integral equation on an open arc. In [8] the authors consider Galerkin method for weakly singular Fredholm integral equations of the second kind using Legendre polynomial basis functions of degree $\leq$ n. The adapted quadrature method and the Sanikidzé's one are used to give a numerical approach to the singular integrals [4, 5, 6, 7, 12]. Reichel in [11] describes a Fourier-Galerkin method as a fast iterative method for solving systems of Fredholm integral equations of the first kind with kernels that have a logarithmic principal part.

The main idea of this work is to propose a numerical treatment for the integral equation of the second kind with logarithmic singular kernel on a piecewise smooth integration path using an adapted cubic approximation and convert this latter into a linear system of algebraic equations, this approximation gives an efficient approach to the analytical solution of weakly singular integral equations.

2000 *Mathematics Subject Classification.* Primary 45D05, 45E05, 45L05; Secondary 65R20.

Key words and phrases. Weakly singular integral, cubic interpolation, Hőlder space and Hőlder condition.

Submitted August 30, 2023.

$$(1)\quad a_0(t_0)\varphi(t_0)+\frac{b_0(t_0)}{\pi i}\int_\Gamma \ln(t-t_0)\varphi(t)dt+\frac{1}{\pi i}\int_\Gamma k(t,t_0)\varphi(t)dt=f(t_0),$$

where Γ is any piecewise smooth closed contour or an open smooth arc [3], t_0 and t are points on Γ, the known functions $a_0(t)$, $b_0(t)$ and $k(t,t_0)$ are defined on Γ and satisfying the Hölder condition $H(\alpha)$, $0<\alpha\leq 1$ [3]. Further, anywhere on Γ we have

$$(2)\qquad a_0(t_0)\neq 0,\ a_0'(t_0)\neq 0 \quad\text{and}\quad a_0'(t_0)+2b_0(t_0)\neq 0.$$

The smooth arc means that the parametric representation of Γ given by $x=x(s)$ and $y=y(s)$; $s\in[s_a,s_b]$ have a continuous derivatives $x'=x'(s)$ and $y'(s)=y'(s)$ not simultaneously null on the interval $[s_a,s_b]$,where $a=(x(s_a),y(s_a))$ and $b=(x(s_b),y(s_b))$ in other words the direction of its tangent in its points varies continuously. The curve Γ is called a simple open arc if $x(s_1)\neq x(s_2)$ and $y(s_1)\neq y(s_2)$ for all $s_1,\ s_2$ in $[s_a,s_b]$ with $s_1\neq s_2$, but if $x(s_a)=x(s_b)$ and $y(s_a)=y(s_b)$ the curve will be called a closed contour, the case where the end s_a,s_b is not in the curve it will be called an open arc.

As it is known, the integral of the dominant part of the above equation (1) exists in the sense of a Cauchy principal value integral for all density φ that satisfies the Holder condition $H(\alpha)$ and also exists for all function $\varphi\in L^2(\Gamma)$.

The present note is divided into two parts. In the first one, we present a formulation of the quadrature formula for the evaluation of weakly singular integral proposed by the author [7], this quadrature formula is based on the adapted cubic approximation of the density $\varphi(t)$.

In the second part, we present the numerical realization of this approximation; also the estimate of the error of the approximation integral was established. Besides, pointwise convergence of the approximate solutions to an exact solution is obtained [7, 9, 10, 12].

A method to proceed is to solve the (W.S.I.E) by numerical means, like the reduction to a system of linear algebraic equations after the use of an appropriate quadrature rule.

2. Quadrature

We denote by t the parametric complex function $t(s)$ of the curve Γ defined by

$$t(s)\ =x(s)+iy(s),\quad a_1\leq s\leq b_1,$$

where $x(s)$ and $y(s)$ are continuous functions on the finite interval of definition $[a_1,b_1]$ and have continuous first derivatives $x'(s)$ and $y'(s)$ never simultaneously null. Let N be an arbitrary natural number, generally we take it large enough and divide the interval $[a_1,b_1]$ into N equal subintervals $I_1,I_2,...,I_N$ by the points

$$s_\sigma=a_1+\sigma\frac{l}{N}\ ,\ l=b_1-a_1\ ,\ \sigma=0,1,2,....,N.$$

Further, we fix a natural number $M > 1$, and divide each of the segments $[s_\sigma, s_{\sigma+1}]$ by the equidistant points

$$s_{\sigma k} = s_\sigma + k\frac{h}{M},\ h = \frac{l}{N},\ k = 0, 1, 2, ..M.$$

In other words, for $M = 3$, we have for each subinterval $[s_\sigma, s_{\sigma+1}]$ the following subdivision

$$[s_\sigma, s_{\sigma+1}] = \{s_\sigma = s_{\sigma 0} < s_{\sigma 1} < s_{\sigma 2} < s_{\sigma 3} = s_{\sigma+1}\}.$$

We introduce the notation

$$t_\sigma = t(s_\sigma),\ t_{\sigma k} = t(s_{\sigma k});\quad \sigma = 0, 1, 2, ..., N;\quad k = 0, 1, 2, 3.$$

Assuming that, for the indices $\sigma, \nu = 0, 1, 2, ..., N-1$, the points t and t_0 belong respectively to the arcs $\overset{\frown}{t_\sigma t_{\sigma+1}}$ and $\overset{\frown}{t_\nu t_{\nu+1}}$ where $\overset{\frown}{t_\mu t_{\mu+1}}$ designates the smallest arc with ends t_μ and $t_{\mu+1}$.

For an arbitrary number $\sigma = 0, 1, 2, ..., N-1$, we define the piecewise cubic Lagrange interpolation polynomial $S_3(\varphi; t, \sigma)$ dependent on φ, t and σ which represents the cubic approximation of the function density $\varphi(t)$ on the subinterval $[t_\sigma, t_{\sigma+1}]$ of the curve Γ. Noting that, the interval $[t_\sigma, t_{\sigma+1}]$ is divided into subintervals $[t_{\sigma k}, t_{\sigma(k+1)}]$ of length $(t_{\sigma(k+1)} - t_{\sigma k}),\ k = 0, 1, 2$. We interpolate the function density $\varphi(t)$ with respect to the values $\varphi(t_{\sigma 0}),\ \varphi(t_{\sigma 1}),\ \varphi(t_{\sigma 2})$ and $\varphi(t_{\sigma 3})$ at the points $t_{\sigma 0},\ t_{\sigma 1},\ t_{\sigma 2}$ and $t_{\sigma 3}$ respectively with a cubic polynomial, given by the following formula.

For $t_\sigma \le t \le t_{\sigma+1}$,

$$\begin{aligned} S_3(\varphi; t, \sigma) = &\frac{(t-t_{\sigma 1})(t-t_{\sigma 2})(t-t_{\sigma 3})}{(t_{\sigma 0}-t_{\sigma 1})(t_{\sigma 0}-t_{\sigma 2})(t_{\sigma 0}-t_{\sigma 3})}\varphi(t_{\sigma 0}) \\ &+ \frac{(t-t_{\sigma 0)})(t-t_{\sigma 2})(t-t_{\sigma 3})}{(t_{\sigma 1}-t_{\sigma 0})(t_{\sigma 1}-t_{\sigma 2})(t_{\sigma 1}-t_{\sigma 3})}\varphi(t_{\sigma 1}) \\ &+ \frac{(t-t_{\sigma 0})(t-t_{\sigma 1})(t-t_{\sigma 3})}{(t_{\sigma 2}-t_{\sigma 0})(t_{\sigma 2}-t_{\sigma 1})(t_{\sigma 2}-t_{\sigma 3})}\varphi(t_{\sigma 2}) \\ &+ \frac{(t-t_{\sigma 0})(t-t_{\sigma 1})(t-t_{\sigma 2})}{(t_{\sigma 3}-t_{\sigma 0})(t_{\sigma 3}-t_{\sigma 1})(t_{\sigma 3}-t_{\sigma 2})}\varphi\left(t_{\sigma 3}\right), \end{aligned} \tag{3}$$

this piecewise cubic interpolating polynomial exists and is unique [1].

For any numbers σ and ν, such that $0 \le \sigma, \nu \le N-1$, we define the continuous functions $U_\sigma(\varphi; t, t_0)$ and$V_{\sigma\nu}(\varphi; t, t_0)$, dependents on $\varphi,\ t$ and t_0 by

$$\begin{aligned} U_\sigma(\varphi; t, t_0) = &\frac{(t-t_{\sigma 1})(t-t_{\sigma 2})(t-t_{\sigma 3})}{(t_{\sigma 0}-t_{\sigma 1})(t_{\sigma 0}-t_{\sigma 2})(t_{\sigma 0}-t_{\sigma 3})}\varphi(t_{\sigma 0})\frac{\ln(t_{\sigma 0}-t_0)}{\ln(t-t_0)} \\ + &\frac{(t-t_{\sigma 0)})(t-t_{\sigma 2})(t-t_{\sigma 3})}{(t_{\sigma 1}-t_{\sigma 0})(t_{\sigma 1}-t_{\sigma 2})(t_{\sigma 1}-t_{\sigma 3})}\varphi(t_{\sigma 1})\frac{\ln(t_{\sigma 1}-t_0)}{\ln(t-t_0)} \\ + &\frac{(t-t_{\sigma 0})(t-t_{\sigma 1})(t-t_{\sigma 3})}{(t_{\sigma 2}-t_{\sigma 0})(t_{\sigma 2}-t_{\sigma 1})(t_{\sigma 2}-t_{\sigma 3})}\varphi(t_{\sigma 2})\frac{\ln(t_{\sigma 2}-t_0)}{\ln(t-t_0)} \\ + &\frac{(t-t_{\sigma 0})(t-t_{\sigma 1})(t-t_{\sigma 2})}{(t_{\sigma 3}-t_{\sigma 0})(t_{\sigma 3}-t_{\sigma 1})(t_{\sigma 3}-t_{\sigma 2})}\varphi\left(t_{\sigma 3}\right)\frac{\ln(t_{\sigma 3}-t_0)}{\ln(t-t_0)} \end{aligned} \tag{4}$$

and

(5)

$$V_{\sigma\nu}(\varphi;t,t_0)=S_3(\varphi;t_0,\nu)\frac{(t-t_{\sigma1})(t-t_{\sigma2})(t-t_{\sigma3})}{(t_{\sigma0}-t_{\sigma1})(t_{\sigma0}-t_{\sigma2})(t_{\sigma0}-t_{\sigma3})}\frac{\ln(t_{\sigma0}-t_0)}{\ln(t-t_0)}$$
$$+S_3(\varphi;t_0,\nu)\frac{(t-t_{\sigma0)})(t-t_{\sigma2})(t-t_{\sigma3})}{(t_{\sigma1}-t_{\sigma0})(t_{\sigma1}-t_{\sigma2})(t_{\sigma1}-t_{\sigma3})}\frac{\ln(t_{\sigma1}-t_0)}{\ln(t-t_0)}$$
$$+S_3(\varphi;t_0,\nu)\frac{(t-t_{\sigma0})(t-t_{\sigma1})(t-t_{\sigma3})}{(t_{\sigma2}-t_{\sigma0})(t_{\sigma2}-t_{\sigma1})(t_{\sigma2}-t_{\sigma3})}\frac{\ln(t_{\sigma2}-t_0)}{\ln(t-t_0)}$$
$$+S_3(\varphi;t_0,\nu)\frac{(t-t_{\sigma0})(t-t_{\sigma1})(t-t_{\sigma2})}{(t_{\sigma3}-t_{\sigma0})(t_{\sigma3}-t_{\sigma1})(t_{\sigma3}-t_{\sigma2})}\frac{\ln(t_{\sigma3}-t_0)}{\ln(t-t_0)},$$

where the functions $U_\sigma(\varphi;t,t_0)$ and$V_{\sigma\nu}(\varphi;t,t_0)$ represent a modified cubic interpolation of the functions $\varphi(t)$ and $\varphi(t_0)$ on the subinterval $[t_\sigma,t_{\sigma+1}]$ and $[t_\nu,t_{\nu+1}]$ respectively of the curve Γ. Define the function $\beta_{\sigma\nu}(\varphi;t,t_0)$ for $t_\sigma\le t\le t_{\sigma+1}$ and $t-t_0\neq 1$,as

$$\beta_{\sigma\nu}(\varphi;t,t_0)=\begin{cases} U_\sigma(\varphi;t,t_0)-V_{\sigma\nu}(\varphi;t,t_0) & \text{for} \quad t\neq t_0\\ 0 & \text{for} \quad t=t_0\end{cases}. \tag{6}$$

Denoting by $\psi_{\sigma\nu}(\varphi;t,t_0)$ the cubic approximation of the density $\varphi(t)$ at the point $t\in[t_\sigma,t_{\sigma+1}]$, $t_0\in[t_\nu,t_{\nu+1}]$ and $0\le\sigma,\nu\le N-1$ by

$$\psi_{\sigma\nu}(\varphi;t,t_0)=\varphi(t_0)+\beta_{\sigma\nu}(\varphi;t,t_0). \tag{7}$$

Using the cubic spline interpolation of the kernel $k(t,t_0)$ and of the density $\varphi(t)$, the regular part of the singular integral equation (1) will be obtained as

$$\begin{aligned} K\varphi(t_0) &= \frac{1}{\pi i}\int_\Gamma k(t,t_0)\varphi(t)dt\\ &\simeq \frac{1}{\pi i}\sum_{\sigma=0}^{N-1}\int_{t_\sigma}^{t_{\sigma+1}}\frac{(t-t_{\sigma1})(t-t_{\sigma2})(t-t_{\sigma3})}{(t_{\sigma0}-t_{\sigma1})(t_{\sigma0}-t_{\sigma2})(t_{\sigma0}-t_{\sigma3})}k(t_{\sigma0},t_0)\varphi(t_{\sigma0})\\ &\quad+\frac{(t-t_{\sigma0)})(t-t_{\sigma2})(t-t_{\sigma3})}{(t_{\sigma1}-t_{\sigma0})(t_{\sigma1}-t_{\sigma2})(t_{\sigma1}-t_{\sigma3})}k(t_{\sigma1},t_0)\varphi(t_{\sigma1})\\ &\quad+\frac{(t-t_{\sigma0})(t-t_{\sigma1})(t-t_{\sigma3})}{(t_{\sigma2}-t_{\sigma0})(t_{\sigma2}-t_{\sigma1})(t_{\sigma2}-t_{\sigma3})}k(t_{\sigma2},t_0)\varphi(t_{\sigma2})\\ &\quad+\frac{(t-t_{\sigma0})(t-t_{\sigma1})(t-t_{\sigma2})}{(t_{\sigma3}-t_{\sigma0})(t_{\sigma3}-t_{\sigma1})(t_{\sigma3}-t_{\sigma2})}k(t_{\sigma3},t_0)\varphi(t_{\sigma3})dt.\\ &= \frac{1}{\pi i}\int_\Gamma k_N(t,t_0)\varphi_N(t)dt\\ &= K_N\varphi_N(t_0). \end{aligned}$$

Let $A\varphi(t_0)$ denote the left side of the equation (1)

$$\begin{aligned} A\varphi(t_0) &= (a_0I + b_0W + K)\varphi(t_0) \\ &= a_0(t_0)\varphi(t_0) + \frac{b_0(t_0)}{\pi i}\int_\Gamma \ln(t-t_0)\varphi(t)dt + \frac{1}{\pi i}\int_\Gamma k(t,t_0)\varphi(t)dt \\ &= a_0(t_0)\varphi(t_0) + \frac{b_0(t_0)}{\pi i}\int_\Gamma \ln(t-t_0)\left(\varphi(t)-\varphi(t_0)+\varphi(t_0)\right)dt \\ &\quad +\frac{1}{\pi i}\int_\Gamma k(t,t_0)\varphi(t)dt \\ &= (a_0I + b_0W + K)\varphi(t_0) \end{aligned}$$

and $A_N\varphi_N(t_0)$ be the adapted quadrature interpolation formula for the operator $A\varphi(t)$ given by

$$\begin{aligned} A_N\varphi_N(t_0) &= (a_0I + b_0W_N + K_N)\varphi_N(t_0) \\ &= a_0(t_0)\varphi_N(t_0) + \frac{b_0(t_0)}{\pi i}\int_\Gamma \ln(t-t_0)\psi_{\sigma\nu}(\varphi;t,t_0)dt \\ &\quad +\frac{1}{\pi i}\int_\Gamma k_N(t,t_0)\varphi_N(t)dt \\ &= a_0(t_0)\varphi_N(t_0) + \frac{b_0(t_0)}{\pi i}\int_\Gamma \ln(t-t_0)\beta_{\sigma\nu}(\varphi;t,t_0)dt \\ &\quad +\frac{b_0(t_0)}{\pi i}\int_\Gamma \ln(t-t_0)\varphi(t_0)dt + \frac{1}{\pi i}\int_\Gamma k_N(t,t_0)\varphi_N(t)dt \\ &= (a_0I + b_0W_N + K_N)\varphi_N(t_0). \end{aligned}$$

We denote by the function $\varphi_N(t)$ the approximate solution of (1) and find it from the equality of the functions $A_N\varphi_N(t_0)$ and $f(t_0)$ at the points $t_{\sigma k},\ \sigma = 0,1,..,N-1;\ k = 0,1,2,3.$

3. Main results

Theorem 3.1. *The weakly singular integral equation of the form* (1) *with the condition* (2) *has a unique solution* $\varphi(t)$ *and an approximate solution* $\varphi_N(t)$ *converges to the solution* $\varphi(t)$ *with the following estimation*

$$|\varphi(t)-\varphi_N(t)| \le \frac{C_1\ln(3MN)}{(3MN)^\alpha} + \frac{C_2}{(MN)^4}; \qquad M,N>1,$$

where the constants C_1 and C_2 *depend only on the curve* Γ *and the Holder constant of the function* φ.

Proof. We can write the integral equation (1) as

$$A\varphi = (a_0I + b_0W + K)\varphi = f,$$

while as an approximating equation in the space $H(\alpha)$ we consider

$$A_N\varphi_N = (a_0I + b_0W_N + K_N)\varphi_N = f.$$

It follows from [7] that, for all $\varphi(t)$ in $H(\alpha)$ we have

$$\|W\varphi - W_N\varphi_N\| \le \frac{C_1\ln(2MN)}{(2MN)^\alpha},$$

and also it is known that

$$\|K\varphi - K_N\varphi_N\| \leq \frac{C_2}{(MN)^4},$$

for all K compact and $\varphi \in H(\alpha)$. It is easy to see that

$$\begin{aligned}
|\varphi - \varphi_N| &= \left|\frac{1}{a_0(t)}\right| \left|b_0(t)\left(W\varphi - W_N\varphi_N\right) + (K\varphi - K_N\varphi_N)\right| \\
&\leq \left|\frac{1}{a_0(t)}\right| \left(\left|b_0(t)\left(W\varphi - W_N\varphi_N\right)\right| + \left|(K\varphi - K_N\varphi_N)\right|\right) \\
&\leq \left|\frac{b_0(t)}{a_0(t)}\right| \left|(W\varphi - W_N\varphi_N)\right| + \left|\frac{1}{a_0(t)}\right| \left|(K\varphi - K_N\varphi_N)\right| \\
&\leq \frac{C_3 \ln(2MN)}{(2MN)^\alpha} + \frac{C_4}{(MN)^4}; \qquad M, N > 1,
\end{aligned}$$

where $C_3 = \sup_{t\in\Gamma} \left|\frac{b_0(t)}{a_0(t)}\right| C_1$ and $C_4 = \sup_{t\in\Gamma} \left|\frac{1}{a_0(t)}\right| C_2$. □

4. Numerical experiments

In this section we describe some of the numerical experiments performed in solving the weakly singular integral equations (1). In all cases, the curve Γ designates the unit circle and we chose the right hand side $f(t)$ in such way that we know the exact solution. This exact solution is used only to show that the numerical solution obtained with our method is correct.

We apply the algorithms described in [4, 5] and [7] to solve (W.S.I.E) and we present results concerning the accuracy of the calculations. In this numerical experiments it is easily to see that the matrix of the system of algebraic equation given by our approximation is invertible, confirmed in [4,7].

In each table, φ represents the exact solution given in the sense of the principal value of Cauchy and φ_N corresponds to the approximate solution produced by our approximation at points values interpolation [5, 6].

Example 1

Consider the weakly singular integral equation,

$$\varphi(t_0) - \int_0^1 \ln|t - t_0|\,\varphi(t)dt = f(t_0)$$

where the function second member $f\,(t_0)$ is given by

$$f(t_0) = t_0 - \frac{1}{2}\left(t_0^2 \ln|t_0| + \left(1 - t_0^2\right)\ln|t_0 - 1| - t_0 - \frac{1}{2}\right)$$

and the exact solution $\varphi(t)$ determined as

$$\varphi(t) = t$$

The approximate solution $\varphi_N\,(t)$ of $\varphi(t)$ is obtained by our modified cubic spline approximation for $N = 32$.

Values of points t	Exact sol φ	App sol φ_N	**Error**	Error [1]	Error [8]
0.0000e+00	0.0000e+00	0.0000e+00	1.162e-06	7.1e-05	2.349e-04
1.2500e-01	1.2500e-01	1.2500e-01	1.969e-09	–	2.349e-04
2.5000e-01	2.5000e-01	2.5000e-01	9.448e-10	4.4e-07	2.349e-04
5.0000e-01	5.0000e-01	5.0000e-01	5.551e-16	7.3e-12	2.349e-04
6.2500e-01	6.2500e-01	6.2500e-01	4.021e-10	–	2.349e-04
7.5000e-01	7.5000e-01	7.5000e-01	9.448e-10	4.4e-07	2.349e-04
1.0000e+00	1.0000e+00	1.0000e+00	1.162e-06	7.1e-05	2.349e-04

Example 2

Consider the weakly singular integral equation,

$$\varphi(t_0) - \int_0^1 \ln|t - t_0|\,\varphi(t)dt = f\,(t_0)$$

where the function second member $f\ (t_0)$ is given by

$$f\,(t_0) = \left(\frac{-1}{(t_0+1)^2} - \frac{\ln 2}{(t_0+1)}\right) + \left(-\frac{(t_0-1)\ln|t_0-1|}{2\,(t_0+1)}\right) + \left(\frac{t_0 \ln|t_0|}{t_0+1}\right)$$

and the exact solution $\varphi(t)$ determined as

$$\varphi(t) = -\frac{1}{(1+t^2)^2}$$

The approximate solution $\varphi_N\,(t)$ of $\varphi(t)$ is obtained by our modified cubic spline approximation for $N = 32$.

Values of points t	Exact sol φ	App sol φ_N	**Error**
0.0000e+00	-1.0000e+00	-9.9999e-01	2.33e-06
1.2500e-01	-7.9012e-01	-7.9012e-01	1.10e-08
2.5000e-01	-6.4000e-01	-6.4000e-0	8.75e-09
5.0000e-01	-4.4444e-01	-4.4444e-01	5.17e-09
6.2500e-01	-3.7869e-01	-3.7869e-01	4.07e-09
7.5000e-01	-3.2653e-01	-3.2653e-01	3.31e-09
1.0000e+00	-2.5000e-01	-2.5000e-01	2.89e-07

Example 3

Consider the weakly singular integral equation,

$$\varphi(t_0) - \int_0^1 \ln|t - t_0|\,\varphi(t)dt = f\,(t_0)$$

where the function second member $f\ (t_0)$ is given by

$$f\,(t_0) = \left(\frac{4}{3}t_0^2 + \frac{1}{6}t_0 - \frac{17}{9}\right) + \left(\frac{1}{3}\left(t_0^3 - 3t_0 + 2\right)\ln|t_0-1|\right) + \left(-\frac{1}{3}\left(t_0^3 - 3t_0\right)\ln|t_0|\right)$$

and the exact solution $\varphi(t)$ determined as

$$\varphi(t) = t^2 - 1$$

The approximate solution $\varphi_N\,(t)$ of $\varphi(t)$ is obtained by our modified cubic spline approximation for $N = 32$.

Values of points t	Exact sol φ	App sol φ_N	**Error**
0.0000e+00	-1.0000e+00	-1.0000e+00	5.49e-09
1.2500e-01	-9.8437e-01	-9.8437e-01	8.48e-09
2.5000e-01	-9.3750e-01	-9.3750e-01	8.62e-09
5.0000e-01	-7.5000e-01	-7.5000e-01	8.26e-09
6.2500e-01	-6.0937e-01	-6.0937e-01	7.73e-09
7.5000e-01	-4.3750e-01	-4.3750e-01	6.73e-09
1.0000e+00	0.0000e+00	-2.3305e-06	2.33e-06

Example 4
Consider the weakly singular integral equation,

$$t_0^2\varphi(t_0) + \frac{t_0}{\pi i}\int_\Gamma \ln(t-t_0)\varphi(t)dt = t_0^2\ln(2t_0+5)$$

where the curve Γ designates the unit circle and the function density $\varphi(t)$ is given by

$$\varphi(t) = \ln(2t_0+5).$$

The approximate solution $\varphi_N(t)$ of $\varphi(t)$ is obtained by our modified cubic spline approximation for $N = 32$.

Values of points t	Exact sol φ	App sol φ_N	**Error**
1.00e+00+0.00e+00i	1.94e+00 0.00e+00i	1.94e+00+ 1.54e-10	1.54e-10
7.07e-01 +7.07e-01i	1.88e+00 +2.17e-01i	1.88e+00 +2.17e-01i	3.55e-10
0.00e+00+1.00e+00i	1.68e+00 +3.80e-01i	1.68e+00 +3.80e-01i	1.70e-09
-7.07e-01 +7.07e-01i	1.34e+00 +3.75e-01i	1.34e+00 +3.75e-01i	2.11e-08
-1.00e+00+0.00e+00i	1.09e+00 +0.00e+00i	1.09e+00 +1.86e-09i	3.04e-08
-7.07e-01 -7.07e-01i	1.34e+00 -3.75e-01i	1.34e+00 -3.75e-01i	4.46e-09
0.00e+00-1.00e+00i	1.68e+00 -3.80e-01i	1.68e+00 -3.80e-01i	4.22e-10
7.07e-01 -7.07e-01i	1.88e+00 -2.17e-01i	1.88e+00 -2.17e-01i	1.60e-10

Example 5
Consider the weakly singular integral equation,

$$2t_0\varphi(t_0) + \frac{1}{\pi i}\int_\Gamma \ln(t-t_0)\varphi(t)dt + \frac{1}{\pi i}\int_\Gamma -\frac{t_0\,(2+t)}{t}\varphi(t)dt = \frac{2\,(t_0+1)}{(t_0+2)}$$

where the curve Γ designates the unit circle and the function density $\varphi(t)$ is given by

$$\varphi(t) = -\frac{1}{t+2}.$$

The approximate solution $\varphi_N(t)$ of $\varphi(t)$ is obtained by our modified cubic spline approximation for $N = 32$.

Values of points t	Exact sol φ	App sol φ_N	**Error**
1.00e+00+0.00e+00i	-3.33e-01 +0.00e+00i	-3.33e-01 -3.08e-10i	5.23e-08
7.07e-01 +7.07e-01i	-3.45e-01 +9.03e-02i	-3.45e-01 +9.03e-02i	5.37e-08
0.00e+00+1.00e+00i	-4.00e-01 +2.00e-01i	-4.00e-01 +2.00e-01i	4.93e-08
-7.07e-01 +7.07e-01i	-5.95e-01 +3.25e-01i	-5.95e-01 +3.25e-01i	1.08e-07
-1.00e+00+0.00e+00i	-1.00e+00 +1.22e-16i	-1.00e+00 +3.90e-07i	4.23e-07
-7.07e-01 -7.07e-01i	-5.95e-01 -3.25e-01i	-5.95e-01 -3.25e-01i	6.05e-08
0.00e+00-1.00e+00i	-4.00e-01 -2.00e-01i	-4.00e-01 -2.00e-01i	5.35e-08
7.07e-01 -7.07e-01i	-3.45e-01 -9.03e-02i	-3.45e-01 -9.03e-02i	5.29e-08

Example 6

Consider the weakly singular integral equation,

$$t_0^2\varphi(t_0) + \frac{t_0}{\pi i}\int_\Gamma \ln(t-t_0)\varphi(t)dt + \frac{1}{\pi i}\int_\Gamma \frac{t_0(t+3)}{t}\varphi(t)dt = \frac{3t_0(t_0+2)}{(t_0+3)}$$

where the curve Γ designates the unit circle and the function density $\varphi(t)$ is given by

$$\varphi(t) = \frac{1}{t+3}.$$

The approximate solution $\varphi_N\,(t)$ of $\varphi(t)$ is obtained by our modified cubic approximation for $N = 32$.

Values of points t	Exact sol φ	App sol φ_N	**Error**
1.00e+00+0.00e+00i	2.50e-01+0.00e+00i	2.50e-01 +5.98e-11i	1.75e-08
7.07e-01 +7.07e-01i	2.60e-01 -4.96e-02i	2.60e-01 -4.96e-02i	1.76e-08
0.00e+00+1.00e+00i	3.00e-01 -1.00e-01i	3.00e-01 -1.00e-01i	1.77e-08
-7.07e-01 +7.07e-01i	3.98e-01 -1.22e-01i	3.98e-01 -1.22e-01i	1.56e-08
-1.00e+00+0.00e+00i	5.00e-01 -3.06e-17i	5.00e-01 -5.50e-09i	1.53e-08
-7.07e-01 -7.07e-01i	3.98e-01 +1.22e-01i	3.98e-01 +1.22e-01i	1.68e-08
0.00e+00-1.00e+00i	3.00e-01 +1.00e-01i	3.00e-01 +1.00e-01i	1.72e-08
7.07e-01 -7.07e-01i	2.60e-01 +4.96e-02i	2.60e-01 +4.96e-02i	1.77e-08

Example 7

Consider the weakly singular integral equation,

$$3t_0\,(t_0+4)\,\varphi(t_0) + \frac{(t_0+2)(t_0-6)}{\pi i}\int_\Gamma \ln(t-t_0)\varphi(t)dt + \frac{1}{\pi i}\int_\Gamma \frac{-2t^2+8t+12}{4t(t^2-t-6)}dt$$
$$= \ -\frac{\left(2t_0^2-68t_0+3t_0^3-84\right)}{2(t_0-3)\,(t_0+2)}$$

where the curve Γ designates the unit circle and the function density $\varphi(t)$ is given by

$$\varphi(t) = \frac{-2t^2+8t+12}{4t(t_0+2)(t_0-3)}.$$

The approximate solution $\varphi_N\,(t)$ of $\varphi(t)$ is obtained by our modified cubic approximation for $N = 32$.

Values of points t	Exact sol φ	App sol φ_N	**Error**
1.00e+00+0.00e+00i	-7.50e-01 +0.00e+00i	-7.50e-01 -1.87e-06i	1.90e-06
7.07e-01 +7.07e-01i	-5.76e-01 +3.43e-01i	-5.76e-01 +3.43e-01i	3.21e-06
0.00e+00+1.00e+00i	-2.10e-01 +5.30e-01i	-2.10e-01 +5.30e-01i	3.72e-06
-7.07e-01 +7.07e-01i	9.68e-02 +4.36e-01i	9.68e-02 +4.36e-01i	2.60e-06
-1.00e+00+0.00e+00i	1.25e-01 +0.00e+00i	1.25e-01 +2.07e-06i	2.27e-06
-7.07e-01 -7.07e-01i	9.68e-02 -4.36e-01i	9.68e-02 -4.36e-01i	2.04e-06
0.00e+00-1.00e+00i	-2.10e-01 -5.30e-01i	-2.10e-01 -5.30e-01i	1.68e-06
7.07e-01 -7.07e-01i	-5.76e-01 -3.43e-01i	-5.76e-01 -3.43e-01i	1.60e-06

References

[1] C. T.H Baker, The Numerical Treatment of Integral Equations. Oxford University Press, Oxford, (1978).

[2] J. Gao and Y. L. Jiang, Trigonometric Hermite wavelet approximation for the integral equations of second kind with weakly singular kernel, J. Comput. Appl. Math. 215 (2008) 242-259.

[3] N.I. Muskhelishvili, Sinoular integral equattion (Noordhoof, Holland) (1963).

[4] M. Nadir, J. Antidze, On the numerical solution of singular integral equations using Sanikidze's approximation, Comp Math in Sc Tech. 10(1), 83-89 (2004).

[5] M. Nadir, B. Lakehali, On The Approximation of Singular Integrals FEN DERGİSİ (E-DERGİ). 2007, 2(2), 236-240.

[6] M. Nadir, Adapted Quadratic Approximation for Singular Integrals, in Journal of mathematical Inequalities 4, (3) pp 423-430 (2010).

[7] M. Nadir, Adapted Quadratic Approximation for Logarithmic kernel Integrals, in Fasciculi mathematici 49 (2012) 75-85.

[8] B. L. Panigrahi, G. Nelakanti, Legendre Galerkin method for weakly singular Fredholm integral equations and the corresponding eigenvalue problem, in J Appl Math Comp 43, (2013)175–197

[9] A. Pedas, G. vainikko, Superconvergence of piecewise polynomial collocations for nonlinear weakly singular integral equations, J. Integral Equations Appl. 9 (1997) 379-406.

[10] S. Prossdorf, J. Saranen and I. H. Sloan, A discrete method for the logarithmic kernel integral equation on an open arc, in 3. Austral. Math. Soc. Ser. B 34 (1993) 401-418.

[11] L. REICHEL, A fast method for solving certain integral equations of the first kind with application to conformal mapping, in J. of Comp and App Math 14 (1986) 125-142.

[12] J. Saranen, The modified quadrature method for logarithmic kernel integral equations on closed curves, in J. Integral Equations Appl. 3(4) (1991) 575-600

[13] M. M. Shamooshakya, P. Assarib and H. Adibic, CAS Wavelet Method for the Numerical Solution of Boundary Integral Equations with Logarithmic Singular Kernels, in International Journal of Mathematical Modelling and Computations 4(4) (2014) 377- 387.

Department of Mathematics, Faculty of Sciences of Bizerte. University of Carthage. Jarzouna. Tunisia.

E-mail address: nadir.mohamedraid@yahoo.com

Department of Mathematics, Institute of Engineering of Bizerte. University of Carthage. Jarzouna. Tunisia

E-mail address: adeljaw2002@yahoo.com